profiles calculated with these in Eqs. (1) are shown as curves 2 and 3 of Fig. 1. That for $E^* = 9$ represents roughly the hydrogen/oxygen system at $p_o = 0.1$ atm; that for $E^* = 0$ is more like a hydrocarbon system, or hydrogen/oxygen at higher density. Note that the distance scale for these would be considerably shortened had the half-pressure point chosen for curve 1 been used instead of $\lambda = \frac{1}{2}$ to set the time unit.

Accuracy comparable to that in the previous work[1] was maintained in the present time-dependent calculations, at the cost of using a net several times finer.

Results

Results are presented in Table 1 and Figs. 2 and 3. The $E^* = 9$ calculation is believed to be near a stability boundary, so that a larger f or larger E^* should stabilize it. For $E^* = 0$, the steady solution is stable at large f, is unstable with small amplitude and high frequency between $f \doteq 1.5$ and $f \doteq 1.25$, and unstable with large amplitude and lower frequency between $f \doteq 1.25$ and $f = 1$. At $f = 1$, an incomplete calculation, carried to a point just beyond the minimum of the valley after the first large peak ($t = 28$ in Fig. 3) suggests that both the period and amplitude are a little larger than at $f = 1.25$.

Discussion

The principal difference between these results and those previously reported is the order of appearance of the high and low-frequency components as f is decreased (for $E^* = 0$). In the previous work the low-frequency components appear first, whereas here the high-frequency components appear first. At $E^* = 9, f = 1$, the oscillation is similar to the high frequency one at $E^* = 0$. From the limited parameter study undertaken, we are inclined to believe that the change in the form of the rate function is more important than the choice of different equation of state parameters in producing this reversal.

Our principal object here is the study of one of the simplest types of instability contained in the equations of reactive flow in the detonation regime. The principal simplifications are the model of the reaction mechanism, the constraint of one-dimensional flow, and the simple rear boundary condition (constant-velocity piston). Although these are such as to preclude any quantitative comparison with experiment, we comment briefly on a few situations in which more realistic one-dimensional calculations of this type may find some application.

The front of an unsupported gas detonation in a tube typically contains transverse waves of complex structure and appreciable amplitude, so that the flow is not one-dimensional. However, a longitudinal period of a sort may be defined, and it may be of some interest to compare it to the one we calculate. In a rectangular tube, the tracks of transverse waves form regular patterns of diamond-shaped cells whose apexes are collision points. Along a line parallel to the tube axis passing through a row of these, the shock pressure history is not unlike that from the unstable one-dimensional calculations. The pressure rises rapidly when two transverse waves collide, then falls below the steady CJ value before being boosted again by the next collision. For the hydrogen/oxygen system of Fig. 1, the measured distance between collisions on such a line[5] is 0.9 cm, about 0.6 half-reaction zone lengths or 14 induction-zone lengths. The calculated long period of our $E^* = 0$ system, in which the induction zone is about three-quarters as long as the half-reaction zone, is twice as long, about 28 induction-zone lengths.

Within a cell of such a detonation, particularly in its latter part where the decay rate is less and the front is more nearly plane,[10] the growth rate of perturbation seen in our calculations suggests that one-dimensional instability might appear within the cell in some systems. Such a phenomenon has not been observed. Rajan[11] has performed a calculation intended to apply to this situation. A fairly realistic reaction model is used, and cylindrical symmetry is assumed, with a piston programmed for the velocity history which would generate a conventional blast wave in the nonreactive case. Although the shock starts at a pressure above the CJ von Neumann-point value, no shock pressure oscillation appears. Either his system is one-dimensionally stable (as is ours for $E^* > 9$) or the potential instability is damped by the strong rarefaction behind the front. Edwards and Meddins[12] have recently observed interferometrically a different type of "longitudinal instability" in the latter part of a detonation cell, consisting of an apparently approximately steady profile in which a large density peak follows the regular reaction zone. This phenomenon does not appear in our calculations, nor, apparently, in the calculations of Rajan referred to previously. If the mechanism that the authors propose as an explanation is correct, our reaction model is too simple to produce this effect.

Finally, there is the observation of periodic density variations in blunt-body flows in detonable gas mixtures near CJ velocity.[13] This is probably caused by nearly one-dimensional instabilities of the type we have calculated here, but strongly reinforced by acoustic vibrations between the bow shock and the body.

References

[1] Fickett, W. and Wood, W. W., "Flow Calculations for One-Dimensional Pulsating Detonations," *The Physics of Fluids*, Vol. 9, No. 5, May 1966, pp. 903–916.

[2] Erpenbeck, J. J., "Stability of Idealized One-Reaction Detonations," *The Physics of Fluids*, Vol. 7, No. 5, May 1964, pp. 684–696.

[3] Fickett, W., Jacobson, J. D., and Wood, W. W., "The Method of Characteristics for One-Dimensional Flow with Chemical Reaction," LA-4269, July 1970, Los Alamos Scientific Lab., Los Alamos, N. Mex.

[4] White, D. R., "Density Induction Times in Very Lean Mixtures of D_2, H_2, C_2H_2, and C_2H_4, with O_2," *Eleventh Symposium (International) on Combustion*, The Combustion Inst., Pittsburgh, Pa., 1967. pp. 147–154.

[5] Strehlow, R. A., Maurer, R. E., and Rajan, S., "Transverse Waves in Detonations. I. Spacing in the Hydrogen-Oxygen System," *AIAA Journal*, Vol. 7, No. 2, Feb. 1969, pp. 323–328.

[6] Strehlow, R. A. and Rubins, P. M., "Experimental and Analytical Study of H_2-Air Reaction Kinetics Using a Standing-Wave Normal Shock," *AIAA Journal*, Vol. 7, No. 7, July 1969, pp. 1335–1344.

[7] Gutman, D., Hardwidge, E. A., Dougherty, F. A., and Lutz, R. W., "Shock-Tube Study of the Recombination Rate of Hydrogen Atoms with Oxygen Molecules," *Journal of Chemical Physics*, Vol. 47, No. 11, Dec. 1967, pp. 4400–4407.

[8] Getzinger, R. W. and Blair, L. S., "Recombination in the Hydrogen-Oxygen Reaction: A Shock-Tube Study with Nitrogen and Water Vapour as Third Bodies," *Combustion and Flame*, Vol. 13, No. 5, May 1969, pp. 271–284.

[9] Ripley, D. L. and Gardiner, W. C., "Shock-Tube Study of the Hydrogen-Oxygen Reaction. II. Role of Exchange Initiation," *Journal of Chemical Physics*, Vol. 44, No. 6, March 1966, pp. 2285–2296.

[10] Crooker, A. J., "Phenomenological Investigation of Low Mode Marginal Planar Detonations," Rept. AAE 69-2, Jan. 1969, Univ. of Illinois, Urbana, Ill.

[11] Rajan, S., "On the Interaction Between Chemical Kinetics and Gasdynamics in the Flow Behind a Cylindrical Detonation Front," Tech. Rept. AAE 70-1, March 1970, Univ. of Illinois, Urbana, Ill.

[12] Edwards, D. H. and Meddins, R. J., "Longitudinal Instabilities in Detonation Waves," *Combustion and Flame*, Vol. 16, No. 3, June 1971, pp. 323–325.

[13] Behrens, H., Struth, W., and Wecken, F., "Studies of Hypervelocity Firings into Mixtures of Hydrogen with Air or Oxygen," *Tenth Symposium (International) on Combustion*, The Combustion Inst., Pittsburgh, Pa., 1965, pp. 245–252.

Constitutive Equations for Bimodulus Elastic Materials

FARHAD TABADDOR*

The B. F. Goodrich Company, Akron, Ohio

I. Introduction

THE bimodulus elastic materials are those which possess different elastic properties in tension and compression. For

Received June 14, 1971. Presented in the Third Canadian Congress of Applied Mechanics, Calgary, Alberta, Canada, May 17–21, 1971.

* Senior Engineering Scientist.

such materials Hook's law of elasticity must be replaced by more general constitutive equations. These equations, however, should reduce to Hook's law of elasticity whenever the stress state is that of either no tension or no compression in the system.

The aforementioned considerations are of special interest in reinforced materials, e.g., the difference in Young's moduli of Boron Epoxy composites in compression and tension has been experimentally observed. The one-dimensional stress-strain relation of textile-cord reinforced rubber exhibits a very sharp bilinearity.[1]

Ambartsumyan[2–7] has suggested a three-dimensional, stress-strain law for isotropic bimodulus materials. He later developed such relations for orthotropic plates.[8] His equations, although physically sound, lack a general theoretical foundation and do not cover the most general case.

In this work a one-dimensional rheological model is proposed, and based on that, the stress-strain laws for the general case of anisotropy are postulated. By a systematic approach those laws are obtained for two important classes of symmetry, namely isotropic and orthotropic cases. Equations for isotropic case are shown to reduce to those of Ambartsumyan, under certain assumptions, thus giving a general theoretical basis as well as the exact assumptions behind those equations.

II. General Equations

The one-dimensional stress-strain relationship for a bimodulus material has the following form

$$\varepsilon = 1/E^+\sigma^+, \quad \text{or,} \quad \varepsilon = 1/E^-\sigma^- \tag{1}$$

ε, σ, and E are strain, stress, and Young's modulus, where plus and minus sign correspond to tension and compression, respectively. Equation (1) can be written in a single form

$$\varepsilon = [1/E^+U(\sigma) + 1/E^-U(-\sigma)]\sigma \tag{2}$$

where U is a step function defined as

$$U(X) = 0 \text{ if } X \leq 0; \quad U(X) = 1 \text{ if } X > 0 \tag{3}$$

The rheological model of Fig. 1, suggested by Eqs. (2) is now proposed. The model consists of two springs with stiffnesses E^+ and E^-. The springs can be activated one at a time depending on the sign of applied stress. A stress-strain law can now be postulated as a direct generalization of the Eq. (2) and the corresponding one-dimensional rheological model. The elastic moduli of the postulated constitutive equations are assumed to be arbitrary functions of the signs of the stresses as follows

$$\varepsilon_{ij} = C_{ijkl}\sigma_{kl} \tag{4}$$

Summation convention is used throughout the paper where

$$C_{ijkl} = C^+_{ijklmn}U[f_{mn}(\sigma_{pq})] + C^-_{ijklmn}U[-f_{mn}(\sigma_{pq})] \tag{5}$$

ε_{ij}, σ_{kl}, and C_{ijkl} are components of the strain tensor, stress tensor, and bimodulus elasticity tensor, respectively. The functions f_{mn} are arbitrary functions of stress-state at this stage.

The constitutive Eq. (4) must however satisfy the laws that are pertinent to constitutive equations of continuum mechanics. It is easily seen that the only law, relevant to the present case, which is not automatically satisfied is the principle of coordinate invariance. This requirement would be met if f_{mn} are isotropic functions of their arguments and hence independent of coordinate rotation, that is

$$f_{mn} = f_{mn}(\textstyle\sum_{\mathrm{I}}, \sum_{\mathrm{II}}, \sum_{\mathrm{III}}) \tag{6}$$

where $\sum_{\mathrm{I}}$, $\sum_{\mathrm{II}}$, and $\sum_{\mathrm{III}}$ are stress tensor invariants. Let us consider the following choice of

$$f_{mn} = 0 \text{ if } m \neq n; \quad f_{mn} = \sigma_{\mathrm{I}}, \sigma_{\mathrm{II}}, \sigma_{\mathrm{III}} \text{ if } m = n \tag{7}$$

where I, II, and III correspond to the directions of principal stresses. We notice that the principal stresses σ_k are the three roots of the following cubic equation

$$\sigma^3 - \textstyle\sum_{\mathrm{I}}\sigma^2 + \sum_{\mathrm{II}}\sigma - \sum_{\mathrm{III}} = 0 \tag{8}$$

and therefore are functions of stress invariants. We conclude that the choice (7) is a permissible one; however, this approach indicates that such a choice constitutes a special form of more general class of bimodulus materials. The above results show that the bimodulus elasticity constants are functions of the sign of principal stresses.

Incorporating the above results and expressing the equations in matrix notation, yields the following:

$$\varepsilon_i = [C^+_{ijk}U(\sigma_k) + C^-_{ijk}U(-\sigma_k)]\sigma_j \qquad i,j,k = 1,2,\ldots,6 \tag{9}$$

where ε_i and σ_j are the conventional engineering strain and stress. The arguments of the step functions are principal stresses and therefore $\sigma_k = 0$ for $k = 4, 5, 6$. It is therefore more convenient to refer the constitutive equations to a coordinate system coinciding with the direction of principal stresses. In that case the stresses corresponding to both k and j are zero for values 4–6. The directions of principal stresses, however, are not generally the same as those of principal strains.

It is the consequence of the preceding form of the constitutive equations that the elasticity constants are functions of principal stresses only. Once the principal stresses are obtained, the corresponding plus or minus superscript constants would be known. The constitutive equations, from there on, behave just like those of classical materials for that fixed state of the stresses.

The moduli are independent of the signs of the shear stresses, even though, the overall behaviour of the material might seem to indicate to the contrary as discussed in Ref. 9.

III. Strain Energy

Bimodulus materials have all the properties of classical elastic materials, in the absence of tension or compression in the system. We now consider the symmetry conditions imposed on elastic constants due to the existence of the strain energy functions in the cases of no tension or no compression.

The strain energy function, expressed in term of stresses, referred to directions of principal stresses, for the case of no compression, has the following form:

$$\begin{aligned} 2W &= C^+_{ijk}\sigma_i\sigma_j U(\sigma_k) \qquad i = 1,2,\ldots,6 \\ \sigma_k, \sigma_j &= 0 \text{ for } k,j = 4,5,6, \quad C^+_{ijk} = 0 \text{ for } k = 4,5,6 \end{aligned} \tag{10}$$

taking the derivative of W with respect to σ_m, yields

$$\begin{aligned} \partial W/\partial\sigma_m = {} & \tfrac{1}{2}C^+_{ijk}(\partial\sigma_i/\partial\sigma_m)\sigma_j U(\sigma_k) + \tfrac{1}{2}C^+_{ijk}\sigma_i(\partial\sigma_j/\partial\sigma_m)U(\sigma_k) \\ & + \tfrac{1}{2}C^+_{ijk}\sigma_i\sigma_j[\partial U(\sigma_k)/\partial\sigma_m] \end{aligned} \tag{11}$$

since $\partial U(\sigma_k)/\partial\sigma_m$ is indeterminate for $\sigma_k = 0$, so the preceding derivative exists only when all the principal stresses are greater than zero; in that case

$$\partial W/\partial\sigma_m = \tfrac{1}{2}C^+_{mjk}\sigma_j U(\sigma_k) + \tfrac{1}{2}C^+_{jmk}\sigma_j U(\sigma_k) \tag{12}$$

on the other hand

$$\partial W/\partial\sigma_m = \varepsilon_m = C^+_{mjk}\sigma_j U(\sigma_k) \tag{13}$$

taking the derivatives of Eqs. (12) and (13) with respect to σ_j, and equating them, results in the following symmetry condition:

$$C^+_{ijk}U(\sigma_k) = C^+_{jik}U(\sigma_k) \text{ if } \sigma_k > 0 \text{ for } k = \mathrm{I, II, III} \tag{14}$$

similarly

$$C^-_{ijk}U(-\sigma_k) = C^-_{jik}U(-\sigma_k) \text{ if } \sigma_k < 0 \text{ for } k = \mathrm{I, II, III} \tag{15}$$

IV. Constitutive Equations for Isotropic Materials

Before proceeding to find the restrictions imposed on constitutive equations due to this class of symmetry, it is relevant to clearly define the meaning of isotropy. An isotropic material has no directional preference with respect to material coordinate, that is to say, that any choice of material coordinate system is irrelevant to the physics of the problem while the material might exhibit different properties in different directions as the consequence of the dependence of the moduli upon stresses. Throughout the rest of this paper we make the following assumptions.

1) The elastic coefficients reduce to those of classical elastic materials in the absence of either compression or tension in the system, however those two sets of elastic coefficients corresponding to the case of no tension or no compression are not numerically the same.

2) The constants related to the coupling of stresses and stress signs are assumed to be negligible compared to those of direct effect that is

$$C_{ijk}^{\mp} = 0 \text{ if } J \neq k$$

By incorporating the aforementioned assumptions together with isotropy requirement for the cases of no tension or no compression and symmetry condition (14) and (15), one would obtain the following:

$$\begin{aligned} C_{ijk}^{\pm} &= k_1^{\pm} \text{ if } i = j = k \\ C_{ijk}^{\pm} &= k_2^{\pm} \text{ if } i \neq j \quad j = k \\ C_{ijk}^{\pm} &= 0 \text{ if } J \neq k \end{aligned} \tag{16}$$

or in a more compact form

$$C_{ijk}^{\mp} = [(K_1^{\mp} - K_2^{\mp})\delta_{ij} + K_2^{\mp}]\delta_{jk} \tag{17}$$

the assumption that $K_1^{+} = K_2^{-}$, would further simplify the constitutive relations to

$$\varepsilon_\alpha = [1/E^+ U(\sigma_\alpha) + 1/E^-(-\sigma_\alpha)]\sigma_\alpha + C(\sigma_\beta + \sigma_\gamma) \tag{18a}$$

$$\varepsilon_\beta = [1/E^+ U(\sigma_\beta) + 1/E^- U(-\sigma_\beta)]\sigma_\beta + C(\sigma_\alpha + \sigma_\gamma) \tag{18b}$$

$$\varepsilon_\gamma = [1/E^+ U(\sigma_\gamma) + 1/E^- U(-\sigma_\gamma)]\sigma_\gamma + C(\sigma_\alpha + \sigma_\beta)$$
$$\alpha = \mathrm{I},\ \beta = \mathrm{II},\ \gamma = \mathrm{III} \tag{18c}$$

where

$$K_1^{+} = 1/E^{+} \quad K_1^{-} = 1/E^{-} \quad K_1^{\mp} = C$$

Equations (18) are equivalent to a single representation of those presented by Ambartsumyan. He deduced the equality of $K_2^{+} = K_2^{-} = -\nu^{-}/E^{-} = -\nu^{+}/E^{+}$ from the symmetry of elasticity constants by incorrect use of strain energy function. It was shown here that the strain energy can be only used in a restricted sense and that does imply a special symmetry condition as given by Eqs. (14) and (15). In a previous paper,[9] it was shown that the matrix of elastic constants for anisotropic case can not be symmetric and such assumption would lead to contradictory results. The above assumption, however, for isotropic case presents no difficulty.

The preceding derivations are referred to principal axes of stress tensor. The equations relative to an arbitrary cartesian system can be obtained by use of well-known transformation relations.

V. Constitutive Equations for Orthotropic Bilinear Materials

As an example we may cite unidirectional reinforced composites. Let the cartesian coordinate system (x, y, z) coincide with the axes of material symmetry. By using the same line of approach as in the isotropic case, the stress-strain relations with respect to the principal directions of the stresses are obtained to be

$$\varepsilon_i = C_{ijk}^{+}\sigma_j U(\sigma_k) + C_{ijk}^{-}\sigma_j U(-\sigma_k) \qquad i,j,k = 1,\ldots,6$$
$$\sigma_j,\ \sigma_k = 0 \text{ for } j,k = 4,5,6 \tag{19}$$

where

$$\begin{aligned} C_{ijk}^{\mp} &= 0 \text{ if } j \neq k \\ C_{ijj}^{\mp} &= C_{jij}^{\mp} \end{aligned} \tag{20}$$

and

$$C_{ijj}^{\mp} = \sum_{m=1}^{6} \sum_{m=1}^{6} a_{mn}^{\mp} q_{mi} q_{mj}$$

where q_{mi} are given in terms of direction cosines of the coordinate transformation.[10] The transformation here is from coordinate of principal stresses to that of (x, y, z) the coordinate of material symmetry.

The constants $a_{mn}^{\mp}$ are 18 elastic constants describing the classical orthotropic properties of the material in the state of no tension and no compression. These two 9 sets of elastic constants are, obviously, to be found from experiments.

The constitutive equations for two-dimensional orthotropic case, together with the proper transformation equations are given in Ref. 9. The latter equations relate the constants of bilinear materials to those of classical elasticity ones. It also has been shown that the matrix of elastic constant is not symmetric for orthotropic case. Some interesting features of composite materials were explained and accounted for by the present theory, these constitutive equations are used to formulate a beam theory for bimodulus elastic materials.[11]

References

[1] Clark, S. K., "The Plane Elastic Characteristics of Cord-Rubber Laminates," *Textile Research Journal*, Vol. 33, No. 4, April, 1963.

[2] Ambartsumyan, S. A., "The Axisymmetric Problem of a Circular Cylindrical Shell Made of Material with Different Strength in Tension and Compression," *Izvestia. Mekhanika*, No. 4, 1965, pp. 77–85; English translation, FTD-HT-23-1055-67.

[3] Ambartsumyan, S. A. and Khachatryan, A. A., "Elasticity for Materials with Different Resistance to Tension and Compression" (in Russian), *Mekhanika Tverdogo Tela*, No. 2, 1966.

[4] Ambartsumyan, S. A., "Equations for a Plane Problem of the Multi-resisting or Multi-modulus Theory of Elasticity" (in Russian), *Isvestia An Armyaskoy SSR, Mekhanika*, No. 2, 1966.

[5] Ambartsumyan, S. A. and Khachatryan, A. A., "Some Problems in the Zero-Moment Theory of Shell Made of Materials with Different Moduli," *SSR, Doklady*, Vol. 43. No. 4, 1966, pp. 198–204; English translation FTD-HT-23-1091-67.

[6] Ambartsumyan, S. A. and Khachatryan, A. A., "A Multi-modulus Elasticity Theory," *Mekhanika Tverdogo Tela*, No. 6, pp. 64–67, 1966; English translation TTF-10, 936, NASA.

[7] Ambartsumyan, S. A. and Khachatryan, A. A., "Zero-Moment Theory for Shells Made of Materials with Diverse Tensile Strength and Compression Strength" (in Russian), AIAA Ref. No. A67-25569.

[8] Ambartsumyan, S. A., "Basic Equations and Relations in the Theory of Elasticity of Anisotropic Bodies with Different Moduli in Tension and Compression," *Mekhanika Tverdogo Tela*, No. 3, pp. 51–61, 1969; trans. 12, Jan. 1970, Aerospace Corp., El Segundo, Calif.

[9] Tabaddor, F., "Two-Dimensional Bi-linear Orthotropic Elastic Materials," *Journal of Composite Materials*, Vol. 3, 1969.

[10] Lekhnitskii, S. G., *Theory of Elasticity of Anisotropic Elastic Body*, Holden-Day, 1963.

[11] Tabaddor, F., "Analysis for Beams Made of Bi-modulus Orthotropic Elastic Materials," to be published.

Bending of a Simply Supported Plate Clamped about a Central Circular Hole

Roy C. Fedotoff* and John F. Carney III†
University of Connecticut, Storrs, Conn.

Introduction

EXACT, closed-form solutions to plate bending problems with irregular boundaries are not always available. The point-matching method yields numerical solutions to boundary value

Received July 8, 1971. The computations were performed in the Computer Center of the University of Connecticut. This facility is supported in part by the National Science Foundation Grant GJ-9.

* Graduate Assistant, Civil Engineering Department; now Consulting Engineer, Bowe, Walsh and Associates, Huntington, N.Y.

† Associate Professor, Civil Engineering Department.